Two-Degree-of-Freedom System, State-Space Method
Revision E

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February 26, 2024

Two-degree-of-freedom System

Consider the damped system in Figure 1.

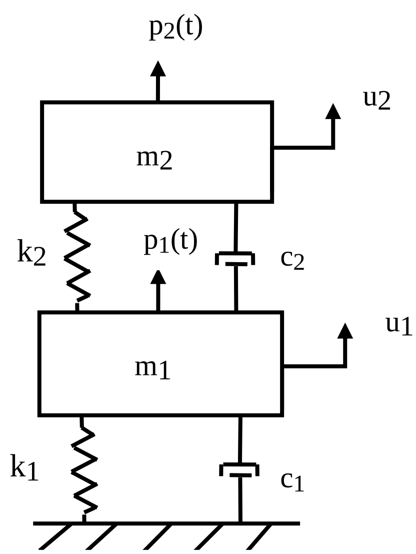


Figure 1. Two-degree-of-freedom System

Structural dynamics systems can be represented in terms of mass, damping and stiffness matrices. Each of these matrices may be coupled depending on the model complexity, degrees-of-freedom, etc. The mass and stiffness matrices in the assembled equation of motion may be uncoupled using the normal modes for the undamped system. This approach gives real natural frequencies and real mode shapes.

Damping effects can be included in forced response analyses by implicitly assuming that the damping matrix can be diagonalized into modal damping coefficients by the undamped modes. But systems with dashpots in general have damping matrices which cannot be uncoupled in this manner.

The state-space method is useful for modal and forced response analysis of systems with discrete dashpot damping. This approach yields complex natural frequencies and mode shapes, with real and imaginary components.

Derivation of Equations of Motion

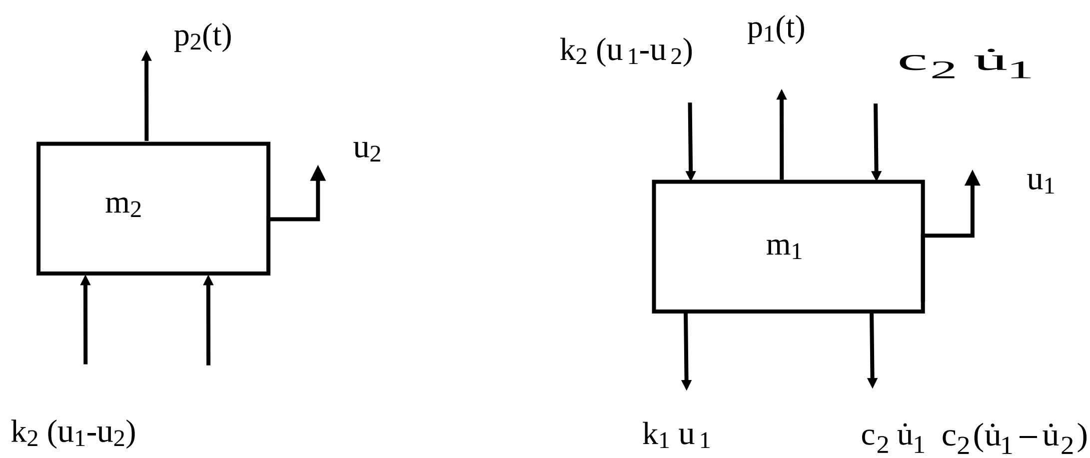


Figure 2. Free-body Diagrams

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ \ddot{u}_2 \tag{1}$$

$$m_2 \ \ddot{u}_2 = p_2(t) + k_2 \ (u_1 - u_2) + c_2 \ (\dot{u}_1 - \dot{u}_2) \tag{2}$$

$$m_2 \ \ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) = p_2(t) \tag{3}$$

Determine the equation of motion for mass 1.

$$\sum \mathrm{F} = \mathrm{m}_1 \ddot{\mathrm{u}}_1 \tag{4}$$

$$\mathrm{m}_1 \ \ddot{\mathrm{u}}_1 = \mathrm{p}_1(\mathrm{t}) - \mathrm{k}_1\mathrm{u}_1 \ - \mathrm{k}_2 \ (\mathrm{u}_1 - \mathrm{u}_2) - \ \mathrm{c}_1\dot{\mathrm{u}}_1 \ - \mathrm{c}_1(\dot{\mathrm{u}}_1 - \dot{\mathrm{u}}_2) \tag{5}$$

$$\mathrm{m}_1 \ \ddot{\mathrm{u}}_1 \ + \mathrm{c}_1\dot{\mathrm{u}}_1 \ + \mathrm{c}_2(\dot{\mathrm{u}}_1 - \dot{\mathrm{u}}_2) + \mathrm{k}_1\mathrm{u}_1 + \mathrm{k}_2(\mathrm{u}_1 - \mathrm{u}_2) \ = \mathrm{p}_1(\mathrm{t}) \tag{6}$$

$$\mathrm{m}_1 \ \ddot{\mathrm{u}}_1 \ + (\mathrm{c}_1 + \mathrm{c}_2)\dot{\mathrm{u}}_1 - \mathrm{c}_2\dot{\mathrm{u}}_2 \ + (\mathrm{k}_1 + \mathrm{k}_2)\mathrm{u}_1 \text{ -}\mathrm{k}_2\mathrm{u}_2 = \mathrm{p}_2(\mathrm{t}) \tag{7}$$

The equations of motion in matrix form are

$$\begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix} \begin{bmatrix} \ddot{\mathrm{u}}_1 \\ \ddot{\mathrm{u}}_2 \end{bmatrix} + \begin{bmatrix} \mathrm{c}_1 + \mathrm{c}_2 & -\mathrm{c}_2 \\ -\mathrm{c}_2 & \mathrm{c}_2 \end{bmatrix} \begin{bmatrix} \dot{\mathrm{u}}_1 \\ \dot{\mathrm{u}}_2 \end{bmatrix} + \begin{bmatrix} \mathrm{k}_1 + \mathrm{k}_2 & -\mathrm{k}_2 \\ -\mathrm{k}_2 & \mathrm{k}_2 \end{bmatrix} \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{p}_1(\mathrm{t}) \\ \mathrm{p}_2(\mathrm{t}) \end{bmatrix} \tag{8}$$

The equations can be represented as

$$\mathrm{M}\ddot{\mathrm{u}} + \mathrm{C}\dot{\mathrm{u}} + \mathrm{K}\mathrm{u} = \mathrm{P} \tag{9}$$

where

$$\mathrm{M} = \begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix}, \quad \mathrm{C} = \begin{bmatrix} \mathrm{c}_1 + \mathrm{c}_2 & -\mathrm{c}_2 \\ -\mathrm{c}_2 & \mathrm{c}_2 \end{bmatrix}, \quad \mathrm{K} = \begin{bmatrix} \mathrm{k}_1 + \mathrm{k}_2 & -\mathrm{k}_2 \\ -\mathrm{k}_2 & \mathrm{k}_2 \end{bmatrix}$$

$$\mathrm{u} = \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix}, \quad \mathrm{P} = \begin{bmatrix} \mathrm{p}_1(\mathrm{t}) \\ \mathrm{p}_2(\mathrm{t}) \end{bmatrix}$$

The dimensions of the mass, damping and stiffness matrices are (N x N), where N is the number of degrees-of-freedom. N = 2 for the example in this paper.

Let

$$\mathrm{z}(\mathrm{t}) = \begin{Bmatrix} \mathrm{u}(\mathrm{t}) \\ \mathrm{v}(\mathrm{t}) \end{Bmatrix}, \quad \dot{\mathrm{z}}(\mathrm{t}) = \begin{Bmatrix} \mathrm{v}(\mathrm{t}) \\ \dot{\mathrm{v}}(\mathrm{t}) \end{Bmatrix} \tag{10}$$

The equations can be expressed in state-space format as

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} v \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ p(t) \end{bmatrix} \tag{11}$$

Premultiply the second row by M^{-1} .

$$\begin{bmatrix} I & 0 \\ 0 & M^{-1}\,M \end{bmatrix} \begin{bmatrix} v \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}\,K & -M^{-1}\,C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1}\,p(t) \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}\,K & -M^{-1}\,C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}\,p(t) \end{bmatrix} \tag{13}$$

Let

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}\,K & -M^{-1}\,C \end{bmatrix} \;,\; \bar{P} = \begin{bmatrix} 0 \\ M^{-1}\,p(t) \end{bmatrix} \tag{14}$$

The A matrix dimension is (2N x 2N).

The A and B matrices are not positive definite. The eigenvalues and vectors are either real or complex, with real and imaginary components. This implies that the modes are not synchronous. There is a phase lag such that different degrees-of-freedom do not simultaneously reach their corresponding peaks and valleys.

The free vibration problem is

$$I\,\dot{z} = A\,z \tag{15}$$

Equation (14) represents a homogeneous set of ordinary differential equations with constant coefficients. The solution can be expressed as

$$z(t) = \psi \exp(\lambda t) \tag{16}$$

$$\dot{z}(t) = \lambda\psi \exp(\lambda t) \tag{17}$$

where λ is a scalar and ψ is a 2N vector

The corresponding generalized eigenvector problem is

$$\lambda \, \mathrm{I} \, \psi \exp(\lambda t) + \mathrm{A} \, \psi \exp(\lambda t) = 0 \quad (18)$$

$$[\lambda \, \mathrm{I} - \mathrm{A} \,] \psi = 0 \quad (19)$$

The solution of this problem yields a set of 2N eigenvalues, i=1, 2, … , 2N.

It also gives corresponding eigenvectors Ψ_i , i=1, 2, … , 2N.

The eigenvalues are found via

$$\det [\lambda \, \mathrm{I} - \mathrm{A}] = 0 \quad (20)$$

The eigenvalues must either be real or complex conjugate pairs because the coefficient matrices are real. Real eigenvalues indicate very high damping leading to overdamped modes. Furthermore, eigenvalues with complex conjugate pairs have corresponding eigenvectors which are complex conjugates.

The eigenvectors can be normalized with respect to the A matrix such that

$$\Psi^{\mathrm{T}} \mathrm{A} \, \Psi = \hat{\lambda} \quad \text{diagonal matrix of eigenvalues} \quad (21)$$

Frequency Response Function Derivation

The non-homogeneous state-space equation can be expressed as

$$\mathrm{I} \, \dot{\bar{z}} = \mathrm{A} \, \bar{z} + \bar{\mathrm{P}} \quad (22)$$

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{z} = \psi \bar{\eta} \quad (23)$$

$$\mathrm{I} \, \psi \dot{\bar{\eta}} = \mathrm{A} \, \psi \bar{\eta} + \bar{\mathrm{P}} \quad (24)$$

Premultiply by the inverse of the eigenvector matrix.

$$\psi^{-1} \, \mathrm{I} \, \psi \, \dot{\bar{\eta}} = \psi^{-1} \mathrm{A} \, \psi \, \bar{\eta} + \psi^{-1} \, \bar{\mathrm{P}} \quad (25)$$

The uncoupled system equation is

$$\mathrm{I}\,\dot{\bar{\eta}} \;=\; \hat{\lambda}\,\bar{\eta} + \psi^{-1}\,\bar{\mathrm{P}} \tag{26}$$

$$\mathrm{I}\,\dot{\bar{\eta}} \;-\; \hat{\lambda}\,\bar{\eta} = \psi^{-1}\,\bar{\mathrm{P}} \tag{27}$$

For a individual coordinate r,

$$\dot{\eta}_r \;-\; \lambda_r \eta_r = (\,\psi^{-1}\,)_r\,\bar{\mathrm{P}} \tag{28}$$

Take the Laplace transform

$$\mathrm{L}\{\dot{\eta}_r \;-\; \lambda_r \eta_r\} = \;\mathrm{L}\{(\,\psi^{-1}\,)_r\,\bar{\mathrm{P}}\} \tag{29}$$

$$s\hat{\eta}_r(s) \;-\eta_r(0) - \;\lambda_r \hat{\eta}_r(s) = \;(\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(s) \tag{30}$$

$$[s \;-\; \lambda_r]\hat{\eta}_r(s) + \eta_r(0) = (\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(s) \tag{31}$$

For zero initial conditions,

$$[s \;-\; \lambda_r]\,\hat{\eta}_r(s) = \;(\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(s) \tag{32}$$

$$\hat{\eta}_r(s) = \;\frac{1}{[s \;-\; \lambda_r]}(\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(s) \tag{33}$$

$$\hat{\eta}(s) = \;\sum_{r=1}^{2N}\frac{1}{[s \;-\; \lambda_r]}(\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(s) \tag{34}$$

Change from the Laplace to the frequency domain with $s = j\omega$.

$$\hat{\eta}(\omega) = \;\sum_{r=1}^{2N}\frac{1}{[\,j\omega \;-\; \lambda_r]}(\,\psi^{-1}\,)_r\,\hat{\mathrm{P}}(\omega) \tag{35}$$

Recall

$$\bar{z} = \psi\bar{\eta} \tag{36}$$

The Fourier transform equivalent for the frequency domain is

$$\hat{Z}(\omega) = \psi\ \hat{\eta}(\omega) \tag{37}$$

$$\hat{Z}(\omega) = \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \\ V_1(\omega) \\ V_2(\omega) \end{Bmatrix} = \sum_{r=1}^{2N} \frac{1}{[j\omega - \lambda_r]}\ \psi\ \ {\psi_r}^{-1}\, \hat{P}(\omega) \tag{38}$$

$$U_i(\omega) = \sum_{r=1}^{2N} \frac{1}{[j\omega - \lambda_r]}\ \psi_{ir}\ \ {\psi_r}^{-1} \begin{bmatrix} 0 \\ M^{-1}\ P(\omega) \end{bmatrix},\ \ i = 1{:}N \tag{39}$$

For N=2,

$$U_i(\omega) = \sum_{r=1}^{4} \frac{1}{[j\omega - \lambda_r]}\ \psi_{ir}\ \ {\psi_r}^{-1} \begin{bmatrix} 0 \\ 0 \\ M^{-1} \begin{bmatrix} P_1(\omega) \\ P_2(\omega) \end{bmatrix} \end{bmatrix},\ \ i = 1,2 \tag{40}$$

Frequency response functions can be derived from the above equation.

References

1. C. Hoen, An Engineering Interpretation of the Complex Eigensolution of Linear Dynamic Systems, IMAC XXIII – Orlando, Florida, 2005.
2. R. Craig & A. Kurdila, Fundamentals of Structural Dynamics, Second Edition, Wiley, New Jersey, 2006.
3. J. Wijker, Mechanical Vibration in Spacecraft Design, Springer, 2003.

APPENDIX A

Example

m_1	2.0 lbm
m_2	1.0 lbm
c1	0.2788 lbf sec/in
c2	0.161 lbf sec/in
k_1	1500 lbf/in
k_2	1000 lbf/in

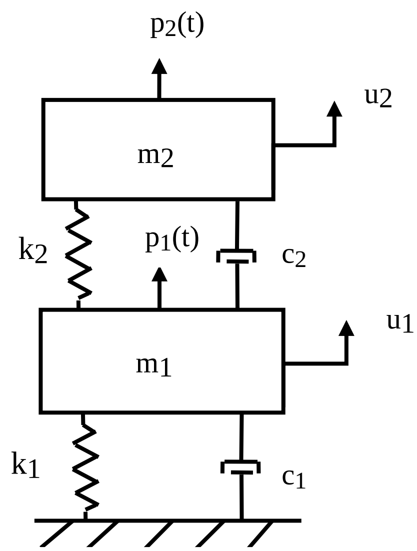


Figure A-1.

The system in Figure A-1 is analyzed via a Matlab script. Note that the mass matrix is divided by 386 inside the Matlab script to convert lbm to lbf sec^2/in.

Mass Matrix 5.181e-03 0.000e+00 0.000e+00 2.591e-03	Damping Coefficient Matrix 4.398e-01 -1.610e-01 -1.610e-01 1.610e-01
Stiffness Matrix 2.500e+03 -1.000e+03 -1.000e+03 1.000e+03	

A =

0.4398	-0.1610	0.0052	0
-0.1610	0.1610	0	0.0026
0.0052	0	0	0
0	0.0026	0	0

Eigenvalues

Complex: -14.13 + -396.1i Mag: 396.3 rad/sec f= 63.08 Hz
Complex: -14.13 + 396.1i Mag: 396.3 rad/sec f= 63.08 Hz
Complex: -59.38 + -841.4i Mag: 843.5 rad/sec f= 134.2 Hz
Complex: -59.38 + 841.4i Mag: 843.5 rad/sec f= 134.2 Hz

lambda1 = -14.13 + -396.1i
lambda2 = -14.13 + 396.1i
lambda3 = -59.38 + -841.4i
lambda4 = -59.38 + 841.4i

Eigenvectors 1 & 2

-5.25e-05 +1.29e-03i	-5.25e-05 -1.29e-03i
-7.74e-05 +2.17e-03i	-7.74e-05 -2.17e-03i
5.10e-01 +2.62e-03i	5.10e-01 -2.62e-03i
8.60e-01 +0.00e+00i	8.60e-01 +0.00e+00i

Eigenvectors 3 & 4

4.55e-05 -7.63e-04i	4.55e-05 +7.63e-04i
-6.38e-05 +9.04e-04i	-6.38e-05 -9.04e-04i
-6.45e-01 +7.05e-03i	-6.45e-01 -7.05e-03i
7.64e-01 +0.00e+00i	7.64e-01 +0.00e+00i

$\psi^{-1} A\, \psi$

$$= \begin{bmatrix} -14.13 + -396.1i & 0 & 0 & 0 \\ 0 & -14.13 + 396.1i & 0 & 0 \\ 0 & 0 & -59.38 + -841.4i & 0 \\ 0 & 0 & 0 & -59.38 + 841.4i \end{bmatrix}$$

$= \hat{\lambda}$ along diagonal

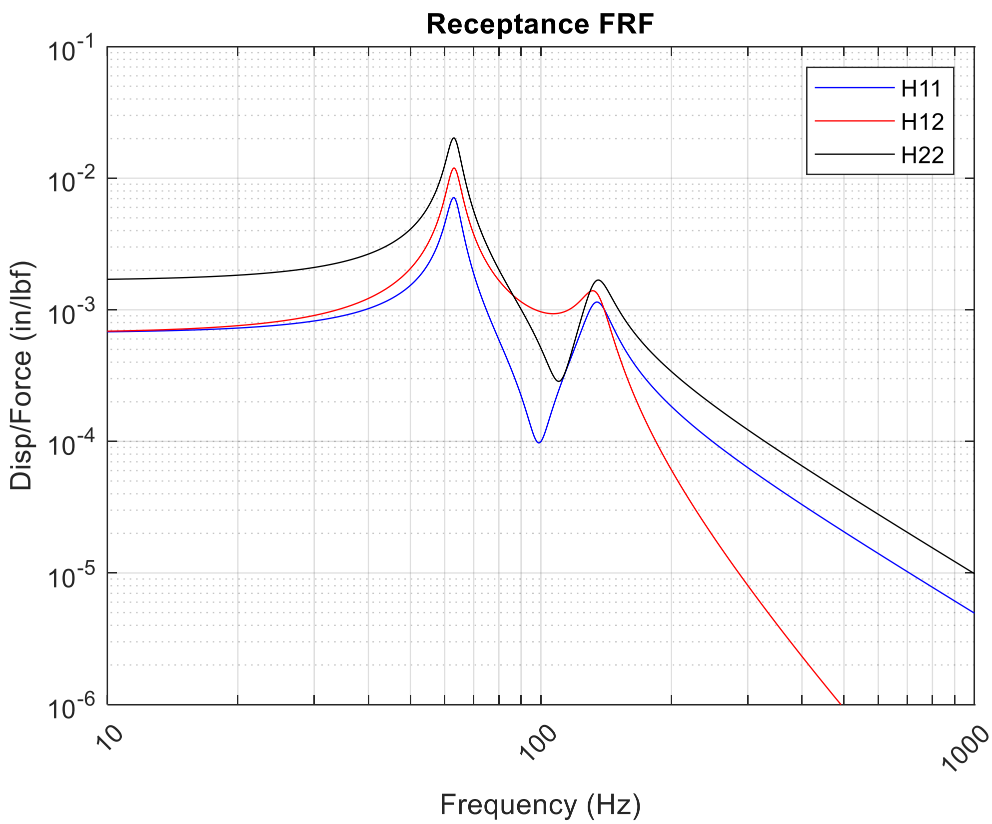
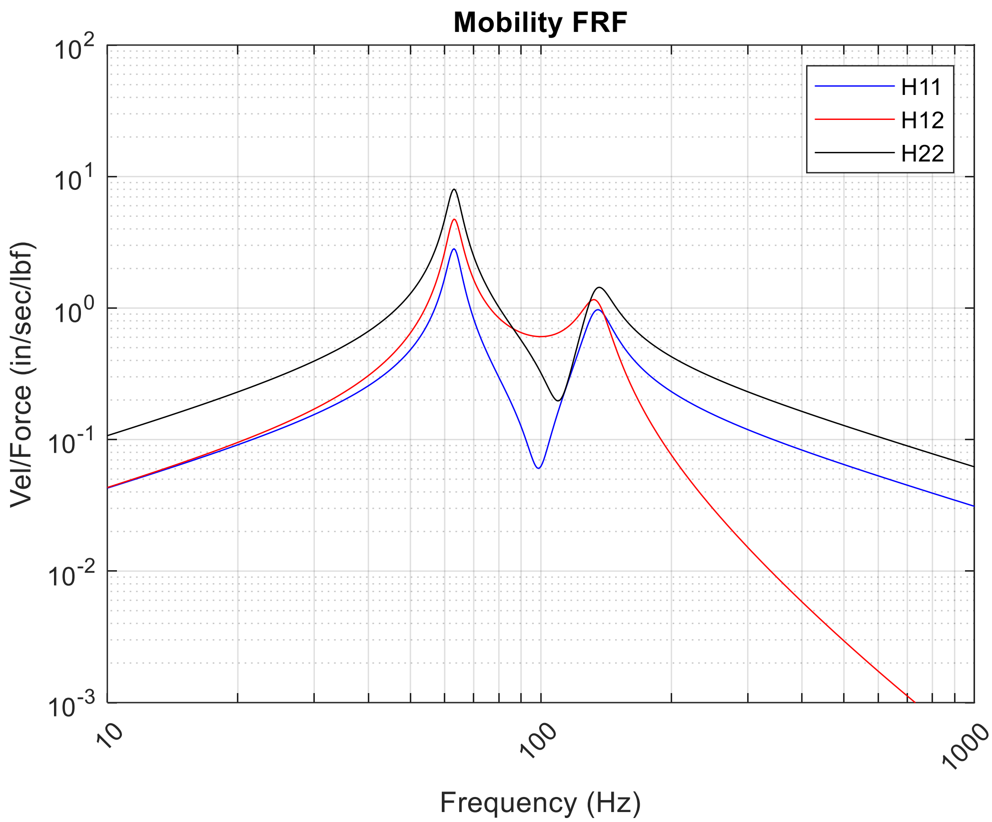
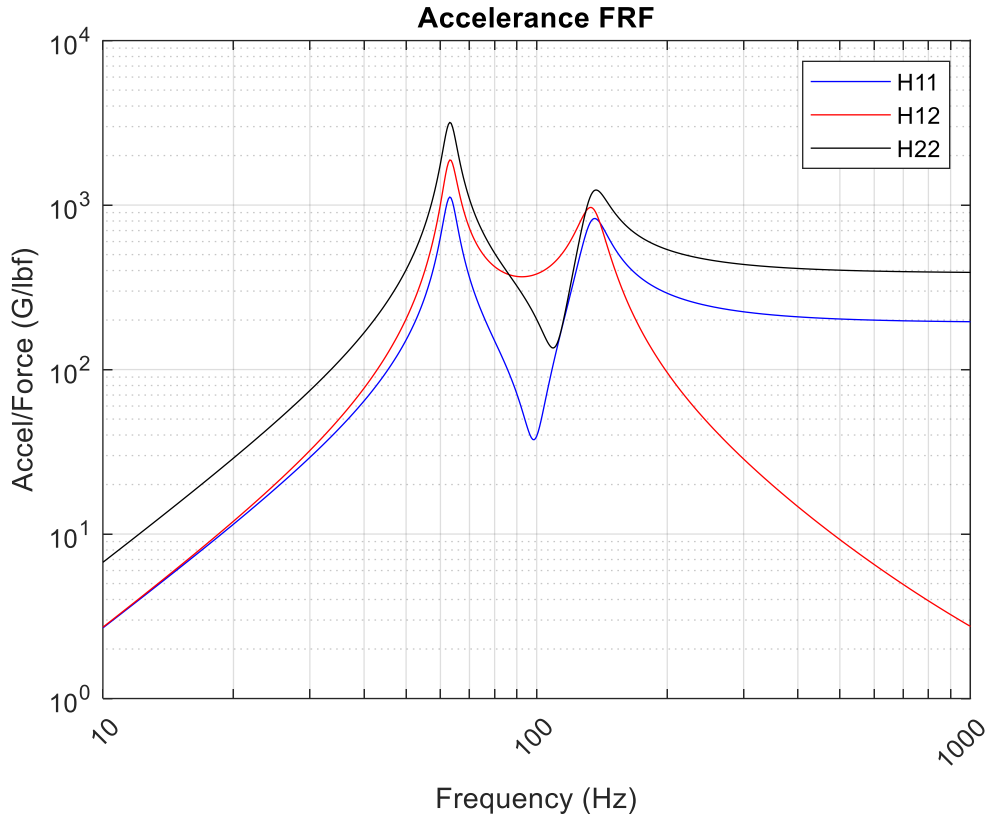


Figure A-2.



Mobility FRF
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
Vel/Force (in/sec/lbf)
10
100
1000
Frequency (Hz)
H11
H12
H22

Figure A-3.



Accelerance FRF
H11
H12
H22
Accel/Force (G/lbf)
10^4
10^3
10^2
10^1
10^0
10
100
1000
Frequency (Hz)

Figure A-4.